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# EQUATIONS OF MOTION OF MECHANICAL SYSTEMS WITH IDEAL ONE-SIDED LINKS

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Routh type differential equations of motion are derived for systems with one-sided links. Derivation is based on the special method of link elimination using a nonsmooth irreversible change of generalized coordinates. Obtained equations were determined over an infinite time interval, and obviate the necessity of resorting to the method of adjustments usually applied in the analysis of such systems. Examples of solutions of problems by the proposed method are given.

Let us consider a mechanical system of the form

$$L(t, q, q^{\cdot}), Q(t, q, q^{\cdot}), f(t, q) \geqslant 0, \quad q, Q \in R^n \tag{1}$$

where $L$ is the Lagrangian, $Q$ is the generalized force, and $f$ is, without loss of generality, a scalar function which defines the generalized link.

Changing variables in that system and passing to new generalized coordinates alters the link equation. Let us aim at finding a change that would eliminate the constraints on the new variables. After such change the equations of motion generally contain singularities and the solutions loose the property of infinite continuation to the right.

We have the problem of finding a change of variables and a descriptive function that would eliminate the constraints on new variables and yield differential equations on the basis of a descriptive function which would be free of singularities and to determine solutions over an infinite time interval. (The term descriptive function denotes a scalar function of time and phase coordinates which permits the derivation of the equation of motion; the Lagrange, Hamilton, Routh, Gibbs, and others functions are examples of descriptive functions).

If function $f(t, q)$ is smoth, there exists a smooth change of variables $q \to r$ such that the inequality that defines the link can be reduced to the form $r_1 \geqslant 0$ (an example of such change is the transformation $r_1 = f(t, q)$, $r_2 = q_2, \ldots, r_n = q_n$, where $r_i$ and $q_i$ are components of vectors $r$ and $q$, respectively). It is obviously necessary to impose on such transformations additional conditions of nondegeneracy in the region of variation of variables in which the motion is considered.

We assume that system (1) possesses these properties. We separate out the component of vector $q$ which appears in the inequality, and introduce for it the special notation $q_1 = s$. For the remaining components we use the notation $(q_2, \ldots, q_n)' = y$. Here and in what follows the prime denotes transposition (all vectors are considered to be column vectors).

In the new notation system (1) assumes the form

$$L\,(t, s, y, s^{\cdot}, y^{\cdot}),\ S\,(t, s, y, s^{\cdot}, y^{\cdot}),\ Y\,(t, s, y, s^{\cdot}, y^{\cdot}),\ s \geqslant 0,\ s \in R^1,\quad y \in R^{n-1} \tag{2}$$

where $S$ and $Y$ are generalized forces corresponding to coordinates $s$ and $y$.

The system Lagrangian $L$ can be assumed not to contain the linear form of generalized velocities, if all forces with a generalized potential are included in the generalized forces. In that case it can be represented as

$$L = \frac{1}{2} \| s^{\cdot}, y^{\cdot\prime} \| \, T \left\| \begin{matrix} s^{\cdot} \\ y^{\cdot} \end{matrix} \right\| + U\,(t, s, y) \tag{3}$$

where $T = T\,(t, s, y)$ is the matrix of kinetic energy and $U$ is the force function.

We decompose matrix $T$ in blocks as follows:

$$T = \left\| \begin{matrix} a & b' \\ b & A^{-1} \end{matrix} \right\|, \quad \begin{matrix} a = T_{11}, & b' = \| T_{12}, \ldots, T_{1n} \| \\ A^{-1} = \{T_{ij}\}, & i, j = 2, \ldots, n \end{matrix} \tag{4}$$

Taking into account the above notation we write the expression for the Lagrangian as

$$L = {}^1/_2\ (as^{\cdot 2} + 2s^{\cdot}b'y^{\cdot} + y^{\cdot\prime}A^{-1}y^{\cdot}) + U\,(t, s, y) \tag{5}$$

We substitute generalized momenta for generalized velocities $y^{\cdot}$ and introduce in the analysis the Routh function. Differentiating (5) we obtain $y^{\cdot}$. We have

$$p = \partial L / \partial y^{\cdot} = s^{\cdot}b + A^{-1}y^{\cdot}, \quad y^{\cdot} = A\,(p - s^{\cdot}b) \tag{6}$$

Using for $y^{\cdot}$ its expression in (6) we obtain for the Routh function

$$R^* = L\,(t, s, y, s^{\cdot}, y^{\cdot}) - p'y^{\cdot} = {}^1/_2\,(a - b'Ab)\ s^{\cdot 2} - {}^1/_2\,p'Ap + s^{\cdot}p'Ab + U\,(t, s, y) \tag{7}$$

We apply the nonsmooth substitution

$$s = |\,x\,| \tag{8}$$

The ambiguity of inverse transformation of (8) does not subsequently lead to any difficulties, since the necessity for it does not arise. This substitution automatically satisfied Eq.(2) of the [system with] link for any $x$ .

With the above substitution taken into account the Routh function [7] assumes the form

$$R\,(t, x, y, x^{\cdot}, p) = R^*\,(t, |\,x\,|, y, x^{\cdot}\,\mathrm{sgn}\,x, p) = R_0 + x^{\cdot}p'\,Ab\,\mathrm{sgn}x \tag{9}$$

$$R_0 = {}^1/_2\,(a - b'Ab)x^{\cdot 2} - {}^1/_2\,p'Ap + U\,(t, |\,x\,|, y)$$

By virtue of (8) the derivative of $|x|$ at zero in the differentiation of $s$ with respect to time is additionally defined in conformity with the expression $s^{\cdot} = x^{\cdot}$ sgn $x$.

Since the generalized force $X$ which corresponds to the new generalized coordinate $x$ is obtained from the equation of balance of powers $Xx^{\cdot} = Ss^{\cdot} = Sx^{\cdot} \operatorname{sgn} x$, hence $X = S \operatorname{sgn} x$.

Since in the system with the new variable effect of link is eliminated, the equations of motion based on the Routh function are of the form of Lagrange equation with variable $x$ and of Hamilton's equation with variables $y$ and $p$. The remarkable property of Routh function (9) is that in spite of the nonsmooth character of substitution (8), the equations of motion determined by that function do not have singularities of the delta-function type. Other descriptive functions (the Lagrange and Hamilton functions) do not have such property, and with them substitution (8) is ineffective.

Let us write down the equations of motion

$$\frac{d}{dt}\frac{\partial R}{\partial x^{\cdot}} - \frac{\partial R}{\partial x} = X; \qquad y^{\cdot} = -\frac{\partial R}{\partial p}, \qquad p^{\cdot} = \frac{\partial R}{\partial y} + Y$$

By virtue of (9) we have

$$\frac{d}{dt}\frac{\partial R}{\partial x^{\cdot}} - \frac{\partial R}{\partial x} = \frac{d}{dt}\frac{\partial R_0}{\partial x^{\cdot}} - \frac{\partial R_0}{\partial x} + \left(\frac{d}{dt}\frac{\partial}{\partial x^{\cdot}} - \frac{\partial}{\partial x}\right) x^{\cdot} p' Ab \operatorname{sgn} x$$

In this formula we calculate the last term separately by differentiating $x^{\cdot} p' Ab \operatorname{sgn} x$ as the generalized function

$$\left(\frac{d}{dt}\frac{\partial}{\partial x^{\cdot}} - \frac{\partial}{\partial x}\right) x^{\cdot} p' Ab \operatorname{sgn} x = \frac{d}{dt}(p'Ab) \operatorname{sgn} x - x^{\cdot} p' \frac{\partial}{\partial x}(Ab) \operatorname{sgn} x$$

in which the terms that could lead to singularities of the delta-function type have cancelled out. The rule of differentiation of complex generalized function $d \operatorname{sgn} x/dt = x^{\cdot} d \operatorname{sgn} x/dx$, applicable when $x^{\cdot}(t)$ is a continuous function was used here.

The continuity of $x^{\cdot}(t)$ is not incompatible with the following equations of motion:

$$\frac{d}{dt}\frac{\partial R_0}{\partial x^{\cdot}} - \frac{\partial R_0}{\partial x} = \left[S - \frac{d}{dt}(p'Ab) + x^{\cdot} p' \frac{\partial}{\partial x}(Ab)\right] \operatorname{sgn} x \tag{10}$$

$$y^{\cdot} = -\frac{\partial R_0}{\partial p} - x^{\cdot} Ab \operatorname{sgn} x, \qquad p^{\cdot} = \frac{\partial R_0}{\partial y} + x^{\cdot}\frac{\partial}{\partial y}(p'Ab) \operatorname{sgn} x + Y$$

which are the sought equations of motion of the considered system with the ideal one-sided link. They have discontinuities of the first kind only, and determine solutions over an infinite time interval, representing phenomena at the beginning, during and at the end of motion constrained by the link.

In the analysis of the solution for the motion at the beginning of restraint by the link (the discontinuity surface of the right-hand sides of system (10)) we distinguish two cases: a) when $x = 0$ and the velocity $x^{\cdot} \neq 0$, and b) when $x = 0$ and $x^{\cdot} = 0$. In the first case a shock is generated in the system when the link constraint begins to act, and all generalized velocities $s^{\cdot}$ and $y^{\cdot}$ become discontinuous at that instant. Motions of system (2) have then the following properties.

1.°. At the instant of shock all generalized momenta that correspond to coordinates not subjected to link constraints (momenta $p$) are continuous. This is obvious from the second equation of system (10). This property was originally established by Apple [1].

2°. The square of velocity of the variable subjected to the link constraint does not become discontinuous at the instant of shock.

This is so because by virtue of the first of Eqs. (10) $x^{\cdot}$ is a continuous function of time, and $s^{\cdot} = x^{\cdot} \operatorname{sgn} x$ implies that $s^{\cdot 2} = x^{\cdot 2}$ is also a continuous function.

3°. The kinetic energy of the system is not discontinuous at the instant of shock.

We introduce in the proof the notation $s_-^{\cdot}$, $y_-^{\cdot}$ and $T_-$ and $s_+^{\cdot}$, $y_+^{\cdot}$ and $T_+$ for the velocities and kinetic energy before and after the shock, respectively. From (5) we then have

$$T_+ = {}^1/_2 \, (as_+^{\cdot 2} + 2s_+^{\cdot} \, b' y_+^{\cdot} + y_+^{\cdot\prime} \, A^{-1} y_+^{\cdot}) \tag{11}$$

In conformity with properties 1° and 2° we have

$$s_+^{\cdot} \, b + A^{-1} y_+^{\cdot} = s_-^{\cdot} b + A^{-1} y_-^{\cdot}, \quad s_+^{\cdot} = - s_-^{\cdot}$$

from which

$$y_+^{\cdot} = 2s_-^{\cdot} \, Ab + y_-^{\cdot} \tag{12}$$

The substitution of (12) into (11) yields

$$T_+ = {}^1/_2 \, [as_-^{\cdot 2} - 2s_-^{\cdot} b' \, (2s_-^{\cdot} Ab + y_-^{\cdot}) + (2s_-^{\cdot} Ab + y_-^{\cdot})' A^{-1} \times$$
$$(2s_-^{\cdot} Ab + y_-^{\cdot})] = {}^1/_2 \, (as_-^{\cdot 2} + 2s_-^{\cdot} b' y_-^{\cdot} + y_-^{\cdot\prime} A y_-^{\cdot}) = T_-$$

When $x = 0$ and $x^{\cdot} = 0$ the system at the beginning of link constraint is free of shock. Subsequently motion under link constrain is possible for some time. The Lagrangian part of the system is then automatically satisfied by $x = 0$, while its Hamiltonian part defines the motion under the link constraint.

Such motion continues as long as the inequality

$$\frac{d}{dt}(p'Ab) + \frac{1}{2} \frac{\partial}{\partial s}(p'Ap) - S - \frac{\partial U}{\partial s} \geqslant 0 \tag{13}$$

is satisfied. The left-hand side of this inequality represents all those terms of the first equation of system (10) that are independent of velocity $x^{\cdot}$. These terms contain the multiplier $\operatorname{sgn} x$. When inequality (13) is satisfied, its left-hand sides represents the link constrain reaction. As soon as this inequality is violated, the system is free of the link constraint.

It should be noted that in the case of motions that initially occur along the link and for which at some instant of time inequality (13) is reversed there is, in addition to the solution of system (10) that corresponds to the true motion, also a solution in which

$x = 0$ for all $t$. This nonuniqueness of solutions for certain initial conditions is the natural consequence of the degeneracy of the substitution $s = | x |$ when $x = 0$. The derivation of solutions for piecewise-continuous systems, and the theorems on the existence and uniqueness of these appear in [2].

As the first example of derivation of Eqs. (10) we consider the simple pendulum with a one-sided link (Fig. 1), i. e. a pendulum in the form of a point-mass suspended on an inextensible string. Its motion is defined here in polar coordinates $(r, y)$, where $y$ is the angle. Distance of the point-mass from the suspension point is bounded by the relation $r \leqslant R$. Using the notation $s = R - r$ we obtain the following system of form (2):

$$L = 1/2m\, [s'^2 + (R - s)^2 y'^2] + mg\, (R - s)\cos y, \; s \geqslant 0$$

from which in conformity with (4) and (9) we have

$$a = m, \; b = 0, \; A^{-1} = m\, (R - s)^2$$
$$R_0 = 1/2 m x'^2 - p^2\, (2m)^{-1}\, (R - |x|)^{-2} + mg\, (R - |x|) \cos y$$

Equation (10) for the pendulum now assumes the form

$$mx'' + G\, (x, y, p) \operatorname{sgn} x = 0, \; G\, (x, y, p) = p^2 m^{-1}\, (R - |x|)^{-3} + mg \cos y \qquad (14)$$
$$y' = pm^{-1}\, (R - |x|)^{-2}, \; p' = - mg\, (R - |x|) \sin y$$

For motions constrained by the link $x = 0$ and $G\, (x, y, p) \geqslant 0$, and the first equation becomes an identity, while the second and third assume the form

$$y' = pm^{-1}R^{-2}, \; p' = - mgR \sin y$$

which are equations of the simple pendulum. At the instant at which $G\, (x, y, p)$ changes its sign the [motion of the ] point ceases to be controlled by the link. The trajectory consisting of sections of motion constrained by the link and free of that constraint satisfies system (14). As previously indicated the trajectory with $x = 0$

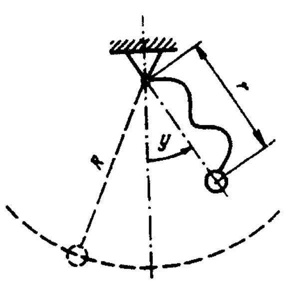


Fig. 1

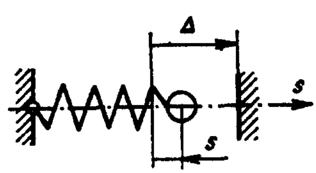


Fig. 2

(for all $t$ ) also satisfies it independently of the link reaction sign.

Let us consider the typical motion with shocks in system (14). For simplicity we select the following intial data (at $t = 0$): $x(0) = x_0$, $0 < x_0 < R$, $x^{\cdot}(0) = 0$, $y(0) = 0$, and $p(0) = 0$. The solution of this system is $y \equiv 0$ and $p \equiv 0$ with $x$ satisfying the equation of the form $x^{\cdot\cdot} + g \operatorname{sgn} x = 0$ which is obtained from (14). It can be shown that its general solution is of the form

$$x(t) = a \int_0^{\varphi} \Pi(\xi)\, d\xi - \frac{\pi^2}{8}, \quad \varphi = \sqrt{\frac{g}{a}}\, t + \theta$$

where function $\Pi(\zeta)$ will be determined below. The coefficients $a$ and $\theta$ are arbitrary constants of integration. Substitution of this solution into $s = |x|$ yields the solution for the variable $s$.

Examples of engineering systems with one-sided links are provided by the so-called vibro-impact systems. The following example shows the effectiveness of the proposed method for obtaining solutions for such systems.

Let us consider the forced oscillations of a harmonic oscillator with a rigid limiter (Fig. 2). The distance between the mass in its equilibrium position on the spring and the limiter face is $\Delta$ (which can be of arbitrary sign). In the theory of vibro-impact systems the equations of motion of such oscillator are usually of the form

$$s^{\cdot\cdot} + hs^{\cdot} + \Omega^2 s = \varepsilon \sin \omega t, \quad s < \Delta$$
$$s^{\cdot}(t_-^*) = -s^{\cdot}(t_+^*), \quad s(t^*) = \Delta$$

In this example we assume the coefficient of restitution to be equal unity.

The above equations of motion are not differential equations, because the conditions at the boundary contain time $t^*$ of the oscillator impact on the limiter, and this is an integral of motion. These equations are, more-over, nonlinear, since they do not satisfy the superposition principle.

Applying the described above method and using the transformation

$$s = \Delta - |x| \tag{15}$$

we obtain for the motion of the oscillator equations of the standard Cauchy form

$$x^{\cdot} = z, \quad z^{\cdot} = -hz - \Omega^2 x + \Omega^2 \Delta \operatorname{sgn} x - \varepsilon \operatorname{sgn} x \sin \omega t \tag{16}$$

which are differential equations and determine the motion for $t \in (-\infty, \infty)$.

We assume the quantities $h$, $\varepsilon$, and $\Delta$ to be small, and substitute in (16) variables, using as the substitution equations the solution (with $h = \varepsilon = \Delta = 0$) of the generating system

$$(x, z) \to (r, \varphi): \ x = r\cos\varphi, \ r = -r\Omega\sin\varphi \ (r > 0) \tag{17}$$

In new variables system (15) assumes the form

$$r^{\cdot} = -hr\sin^2\varphi - \Omega\Delta M(\varphi)\sin\varphi + \varepsilon\Omega^{-1} M(\varphi)\sin\psi\sin\varphi \tag{18}$$
$$\varphi^{\cdot} = \Omega - h\sin\varphi\cos\varphi - \Omega r^{-1}\Delta M(\varphi)\cos\varphi + \varepsilon (r\Omega)^{-1} M(\varphi)\sin\psi\cos\varphi$$

$$\psi^{\cdot} = \omega \quad (M(\varphi) = \operatorname{sgn} \cos\varphi)$$

which is standard for systems with one slow variable $r$ and two fast phases $\varphi$ and $\psi$. We shall analyze that system by the method of averaging. Since the system has discontinuities of the first kind, the use of the averaging method is admissible [3]. Let us consider a resonance of the arbitrary form $n\omega - m\Omega = \chi$ in which $n$ and $m$ are integers and $\chi$ is a small quantity (frequency difference).

In conformity with the procedure of resonance analysis we introduce the slow phase $\theta = n\psi - m\varphi$ from which

$$\psi = n^{-1}(m\varphi + \theta) \tag{19}$$

Substituting (19) into (18) and averaging with respect to $\varphi$ for $n = 1$, $m = 2k$ and $k = 1, 2, \ldots$, we obtain

$$r^{\cdot} = -br + a\cos\theta, \quad \theta^{\cdot} = \chi + cr^{-1} - ar^{-1}\sin\theta$$

$$b = \frac{h}{2}, \quad a = \frac{-4k\varepsilon(-1)^k}{\pi\Omega(4k^2 - 1)}, \quad c = \frac{4k\Omega\Delta}{\pi} \tag{20}$$

For other values of $n$ and $m$ there are no resonances.

The solution of equations of the stationary mode in accordance with (20) is of the form

$$r_0' = -\frac{c\chi}{b^2 + \chi^2} \pm \frac{\sqrt{a^2(b^2 + \chi^2) - b^2c^2}}{b^2 + \chi^2}, \quad \cos\theta_0 = \frac{b}{a}r_0, \tag{21}$$

$$\sin\theta_0 = \frac{\chi}{a}r_0 + \frac{c}{a}$$

Since $r_0 > 0$, hence $c\chi < 0$, and consequently, when $\Delta > 0$, a stationary mode exists in the preresonance region $(n\omega - m\Omega < 0)$ and when $\Delta < 0$, it exists beyond the resonance region $(n\omega - m\Omega > 0)$. One more condition of existence of the resonance mode is, obviously, of the form $a^2(b^2 + \chi^2) - b^2c^2 \geqslant 0$ or in terms of input parameters

$$\varepsilon^2(4k^2 - 1)^{-2} \geqslant \pi^2\Omega^4\Delta^2h^2(h^2 + 4\chi^2)^{-1}$$

This condition shows that resonaces of the form $n = 1$, $m = 2k$, and $k = 1, 2, \ldots$ exist when $h = 0$ or $\Delta = 0$. But when $\Delta h \neq 0$, no resonances exist after the number $k$ has reached a certain value.

Let us investigate the stability of the obtained solutions (21). Taking into account Eqs. (21) of the stationary mode, for the characteristic equation of the variational system corresponding to system (20) we obtain an equation of the form

$$\lambda^2 + 2b\lambda + b^2 + \chi(\chi r_0 + c)r_0^{-1} = 0$$

which yields the necessary and sufficient conditions of asymptotic stability of solution (21)

$$b > 0, \; r_0 > -c\chi\,(b^2 + \chi^2)^{-1}$$

Comparing the second of these conditions with (21), we conclude that the branches of the amplitude-frequency characteristic that correspond to the upper sign in (21) are stable.

Bearing in mind (15), (17), and (19), we write down the solution in terms of the original variable $s$ ($r_0$ and $\theta$ are determined by formula (21))

$$s = \Delta - r_0 \,|\cos (2k)^{-1} (\omega t - \theta_0)\,|$$

This solution is asymptotically close to the exact one when $\Delta, h, \varepsilon \to 0$.

Let us compare this solution with the exact solution presented, for instance, in [4] in the form

$$a = -a_1(\zeta)\, r \pm a_2(\zeta)\, F/k, \quad a_1 = \frac{\sin^2 \pi\zeta/2}{\cos \pi\zeta}, \quad a_2 = \frac{\zeta^2 \cos^2 \pi\zeta/2}{(1-\zeta^2)\cos \pi\zeta} \tag{22}$$

which corresponds to the principal resonance $m = 2$ with $h = 0$.

The relation between the notation used in the last formula and that used here is defined as follows:

$$a = r_0/2, \; \zeta = (\omega - \chi)/(2\omega), \; r = \Delta, \; F/k = 4\varepsilon\,(\omega - \chi)^{-2}$$

Comparison of solution (21) in the indicated notation ($h = 0$) with the exact result shows that in both cases $a$ is the linear form of $r$ and of static displacement $F/k$ of the form (22), with

$$a_1 = \frac{2\zeta}{\pi(1-2\zeta)}, \quad a_2 = \frac{2\zeta}{3\pi(1-2\zeta)}$$

in the case of the approximate solution.

The relative errors of the approximately calculated coefficients $a_1$ and $a_2$ in function $\zeta$ are tabulated below

| $\zeta$ | 0.2 | 0.3 | 0.4 | 0.45 | 0.55 | 0.6 | 0.7 | 0.8 | 1 |
|---|---|---|---|---|---|---|---|---|---|
| $\Delta a_{,1}$ % | 79.8 | 36.2 | 13.9 | 6.3 | 5.3 | 9.8 | 17.5 | 24.1 | 36.3 |
| $\Delta a_{,2}$ % | 51.8 | 19.2 | 5.2 | 1.7 | 0.2 | 1.2 | 10.2 | 34.8 | — |

Since $\zeta = 0.5$ corresponds to the exact resonance tuning, these figures show the reasonably high accuracy of the approximate computations.

Let us now consider one more type of one-sided link. Let system (1) be specified in the form

$$L\,(t, q, q^{\cdot}),\; Q\,(t, q, q^{\cdot}),\; e_1 \geqslant f\,(t, q) \geqslant e_2,\; q, Q \in R^n$$

If $f(t, q)$ is smooth, then as previously, this system can be reduced to the form

$$L(t, s, y, s^{\cdot}, y^{\cdot}),\ S(t, s, y, s^{\cdot}, y^{\cdot}),\ Y(t, s, y, s^{\cdot}, y^{\cdot})$$
$$\pi/2 \geqslant s \geqslant -\pi/2,\ s \in R^1,\ y \in R^{h-1}$$

We introduce function

$$M(x) = \operatorname{sgn} \cos x, \quad \Pi(x) = \int_0^x \operatorname{sgn} \cos x \, dx$$

and then effect the nonsmooth substitution $s = \Pi(x)$.

The subsequent reasoning is analogous to the previous, except that in all formulas and equations $\Pi(x)$ is to be substituted for $|x|$, and $M(x)$ for $\operatorname{sgn} x$. Thus, for example, the equations of motion of the system with the indicated link are of the form (10) after the above substitution.

Ideas on the extension of the obtained here results to the case of nonideal one-sided links can be found in [5, 6].

We note in conclusion that the use of nonsmooth substitutions in the two considered cases is important. For instance, if $s = x^2$ is used instead of $s = |x|$, then for any descriptive functions the differential equations of motion determine the motion only up to the first instance of the link constraint becoming effective.

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